

# ON NON-KÄHLER COMPACT COMPLEX MANIFOLDS WITH BALANCED AND ASTHENO-KÄHLER METRICS

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ABSTRACT. In this note we construct, for every  $n \geq 4$ , a non-Kähler compact complex manifold  $X$  of complex dimension  $n$  admitting a balanced metric and an astheno-Kähler metric which is in addition  $k$ -th Gauduchon for any  $1 \leq k \leq n - 1$ .

## 1. INTRODUCTION

Let  $X$  be a compact complex manifold of complex dimension  $n$ , and let  $F$  be a Hermitian metric on  $X$ . It is well-known that the metric  $F$  is called *balanced* if the Lee form vanishes, equivalently the form  $F^{n-1}$  is closed. If  $\partial\bar{\partial}F^{n-2} = 0$ , then the Hermitian metric  $F$  is said to be *astheno-Kähler*. Balanced metrics are studied by Michelsohn in [15], and the class of astheno-Kähler metrics is considered by Jost and Yau in [13] to extend Siu's rigidity theorem to non-Kähler manifolds. This note is motivated by a question in the paper [16] by Székelyhidi, Tosatti and Weinkove about the existence of examples of non-Kähler compact complex manifolds admitting both balanced and astheno-Kähler metrics. Recently, two examples, in dimensions 4 and 11, are constructed by Fino, Grantcharov and Vezzoni in [4]. Our goal is to present examples in any complex dimension  $n \geq 4$ . Moreover, we show that our astheno-Kähler metrics satisfy the stronger condition of being  $k$ -th Gauduchon for every  $1 \leq k \leq n - 1$ .

When the Lee form is co-closed, equivalently  $F^{n-1}$  is  $\partial\bar{\partial}$ -closed, the Hermitian metric  $F$  is called *standard* or *Gauduchon*. By [10] there is a Gauduchon metric in the conformal class of every Hermitian metric on  $X$ . Fu, Wang and Wu introduce and study in [9] the following generalization of Gauduchon metrics. Let  $k$  be an integer such that  $1 \leq k \leq n - 1$ , a Hermitian metric  $F$  on  $X$  is called  *$k$ -th Gauduchon* if  $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$ .

By definition,  $(n - 1)$ -th Gauduchon metrics are the usual Gauduchon metrics. Astheno-Kähler metrics are particular examples of  $(n - 2)$ -th Gauduchon metrics, and any *pluriclosed* (SKT) metric, i.e. a metric satisfying  $\partial\bar{\partial}F = 0$ , is in particular 1-st Gauduchon.

In [9] a unique constant  $\gamma_k(F)$  is associated to any Hermitian metric  $F$  on  $X$ . This constant is invariant by biholomorphisms and depends smoothly on  $F$ . Moreover, it is proved that  $\gamma_k(F) = 0$  if and only if there exists a  $k$ -th Gauduchon metric in the conformal class of  $F$ .

On a compact complex surface any Hermitian metric is automatically astheno-Kähler, and the balanced condition is the same as the Kähler one. In complex dimension  $n = 3$  the notion of astheno-Kähler metric coincides with that of SKT metric.

SKT or astheno-Kähler metrics on a compact complex manifold  $X$  of complex dimension  $n \geq 3$  cannot be balanced unless they are Kähler (see [1, 14]). If the Lee form is exact, then the Hermitian structure is conformally balanced. By [5, 11] a conformally balanced SKT or astheno-Kähler metric whose Bismut connection has (restricted) holonomy contained in  $SU(n)$  is necessarily Kähler. Similar results for 1-st Gauduchon metrics are proved in [6]. Ivanov and Papadopoulos [12] have extended these results to any generalized  $k$ -th Gauduchon metric, for  $k \neq n - 1$ .

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2010 *Mathematics Subject Classification*. 53C55; 32J27, 53C15.

*Key words and phrases*. Complex manifold; astheno-Kähler metric; balanced metric.

A recent conjecture in [7] asserts that if  $X$  has an SKT metric and another metric which is balanced, then  $X$  is Kähler. By a result of Chiose [3] a manifold in the Fujiki class  $\mathcal{C}$  has no SKT metrics unless it is Kähler. In [8] the conjecture is studied on the class of *complex nilmanifolds*  $X = (\Gamma \backslash G, J)$ , i.e. on compact quotients of simply-connected nilpotent Lie groups  $G$  by uniform discrete subgroups  $\Gamma$  endowed with an invariant complex structure  $J$ . In this note we construct, for every  $n \geq 4$ , a non-SKT complex nilmanifold  $X$  of complex dimension  $n$  admitting a balanced metric and an astheno-Kähler metric which additionally satisfies the stronger condition of being  $k$ -th Gauduchon for every  $1 \leq k \leq n - 1$ .

## 2. GENERALIZED GAUDUCHON METRICS ON COMPLEX NILMANIFOLDS

We first prove the following general result.

**Proposition 2.1.** *Let  $X$  be a compact complex manifold of complex dimension  $n \geq 3$ , and  $F$  any Hermitian metric on  $X$ . For any integer  $k$  such that  $1 \leq k \leq n - 1$ , we have*

$$(1) \quad \int_X \partial \bar{\partial} F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2} \int_X \partial \bar{\partial} F \wedge F^{n-2}.$$

*Proof.* The equality (1) is trivial for  $k = 1$  and for  $k = n - 1$ . Let us then suppose that  $2 \leq k \leq n - 2$ . By induction one has  $\partial F^k = k \partial F \wedge F^{k-1}$  and  $\bar{\partial} F^k = k \bar{\partial} F \wedge F^{k-1}$ . Therefore,

$$(2) \quad \partial \bar{\partial} F^k \wedge F^{n-k-1} = k \partial \bar{\partial} F \wedge F^{n-2} + k(k-1) \partial F \wedge \bar{\partial} F \wedge F^{n-3}.$$

On the other hand,

$$\begin{aligned} \partial \bar{\partial} F^k \wedge F^{n-k-1} &= d \left( \bar{\partial} F^k \wedge F^{n-k-1} \right) + \bar{\partial} F^k \wedge \partial F^{n-k-1} \\ &= d \left( \bar{\partial} F^k \wedge F^{n-k-1} \right) - k(n-k-1) \partial F \wedge \bar{\partial} F \wedge F^{n-3}, \end{aligned}$$

so we get

$$\partial F \wedge \bar{\partial} F \wedge F^{n-3} = \frac{-1}{k(n-k-1)} \left[ \partial \bar{\partial} F^k \wedge F^{n-k-1} - d \left( \bar{\partial} F^k \wedge F^{n-k-1} \right) \right].$$

Now, if we substitute this expression in (2) we have

$$\partial \bar{\partial} F^k \wedge F^{n-k-1} = k \partial \bar{\partial} F \wedge F^{n-2} - \frac{k-1}{n-k-1} \partial \bar{\partial} F^k \wedge F^{n-k-1} + \frac{k-1}{n-k-1} d \left( \bar{\partial} F^k \wedge F^{n-k-1} \right),$$

which leads to

$$(n-2) \partial \bar{\partial} F^k \wedge F^{n-k-1} = k(n-k-1) \partial \bar{\partial} F \wedge F^{n-2} + (k-1) d \left( \bar{\partial} F^k \wedge F^{n-k-1} \right).$$

By Stokes theorem we arrive at (1).  $\square$

Next we apply the previous proposition to homogeneous compact complex manifolds  $X$ , of complex dimension  $n$ , endowed with an *invariant* Hermitian metric  $F$ . We recall that in [6, Lemma 4.7] the following duality result is proved: for each  $k = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$ , the Hermitian metric  $F$  is  $k$ -th Gauduchon if and only if it is  $(n-k-1)$ -th Gauduchon. As a consequence of Proposition 2.1, the relation among these metrics turns out to be stronger:

**Proposition 2.2.** *Let  $F$  be an invariant Hermitian metric on a homogeneous compact complex manifold  $X$  of complex dimension  $n \geq 3$ , and let  $k$  be an integer such that  $1 \leq k \leq n - 2$ . Then,*

- (i)  $F$  is always Gauduchon, and
- (ii) if  $F$  is  $k$ -th Gauduchon for some  $k$ , then it is  $k$ -th Gauduchon for any other  $k$ .

*Proof.* For any invariant Hermitian metric  $F$  and any  $1 \leq k \leq n-2$ , the real  $(n, n)$ -form  $\frac{i}{2} \partial \bar{\partial} F^k \wedge F^{n-k-1}$  is proportional to the volume form  $F^n$ , hence

$$(3) \quad \frac{i}{2} \partial \bar{\partial} F^k \wedge F^{n-k-1} = C_{F,k} F^n,$$

for some constant  $C_{F,k} \in \mathbb{R}$  (notice that  $C_{F,k}$  is a multiple of the constant  $\gamma_k(F)$  in [9]).

If  $k = n-1$  then  $C_{F,n-1} = 0$ , i.e.  $F$  is Gauduchon, because otherwise the form  $F^n$  would be exact. Now, let  $k$  be such that  $1 \leq k \leq n-2$ . From (1) and (3) we get

$$C_{F,k} \int_X F^n = \frac{i}{2} \int_X \partial \bar{\partial} F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2} \frac{i}{2} \int_X \partial \bar{\partial} F \wedge F^{n-2} = \frac{k(n-k-1)}{n-2} C_{F,1} \int_X F^n,$$

that is,  $\left(C_{F,k} - \frac{k(n-k-1)}{n-2} C_{F,1}\right) \int_X F^n = 0$ . Therefore,

$$(4) \quad C_{F,k} = \frac{k(n-k-1)}{n-2} C_{F,1},$$

for any  $k$  such that  $1 \leq k \leq n-2$ . Hence, if  $F$  is  $k$ -th Gauduchon for some  $k$ , then  $C_{F,k} = 0$  and by (4) we get  $C_{F,1} = 0$ . Using again (4) we conclude that  $C_{F,k} = 0$  for any other  $k$ , i.e.  $F$  is  $k$ -th Gauduchon for any  $1 \leq k \leq n-2$ .  $\square$

**Corollary 2.3.** *Let  $X$  be a homogeneous compact complex manifold of complex dimension  $n \geq 3$  and let  $F$  be an invariant Hermitian metric on  $X$ . If  $F$  is SKT or astheno-Kähler, then  $F$  is  $k$ -th Gauduchon for any  $1 \leq k \leq n-1$ .*

**Theorem 2.4.** *For each  $n \geq 4$ , there is a non-Kähler compact complex manifold  $X$  of complex dimension  $n$  admitting a balanced metric  $\tilde{F}$  and an astheno-Kähler metric  $F$  which is additionally  $k$ -th Gauduchon for any  $1 \leq k \leq n-1$ .*

*Proof.* We will construct such an  $X$  using the class of complex nilmanifolds. Let  $(a_1, \dots, a_{n-1}) \in (\mathbb{R} \setminus \{0\})^{n-1}$ , and let  $\{\omega^j\}_{j=1}^n$  be a basis of forms of type  $(1,0)$  satisfying

$$(5) \quad d\omega^1 = \dots = d\omega^{n-1} = 0, \quad d\omega^n = \sum_{j=1}^{n-1} a_j \omega^{j\bar{j}}.$$

(See Remark 2.5 below for more details.) We impose the “canonical” metric  $\tilde{F} = \frac{i}{2}(\omega^{1\bar{1}} + \dots + \omega^{n\bar{n}})$  to be balanced, i.e.  $d\tilde{F}^{n-1} = 0$ . This condition is equivalent to

$$(6) \quad a_1 + \dots + a_{n-1} = 0.$$

Let us now consider a generic “diagonal” metric

$$(7) \quad F = \frac{i}{2}(b_1 \omega^{1\bar{1}} + \dots + b_{n-1} \omega^{n-1\bar{n-1}}) + \frac{i}{2} \omega^{n\bar{n}},$$

where  $b_1, \dots, b_{n-1} \in \mathbb{R}^+$ .

Let  $r \leq n-1$ . We denote  $A_r = a_1 \omega^{1\bar{1}} + \dots + a_r \omega^{r\bar{r}}$  and  $B_r = b_1 \omega^{1\bar{1}} + \dots + b_r \omega^{r\bar{r}}$ . Hence, in (5) and (7) we can write  $d\omega^n = A_{n-1}$  and  $F = \frac{i}{2} B_{n-1} + \frac{i}{2} \omega^{n\bar{n}}$ .

Let us calculate  $\partial \bar{\partial} F^{n-2}$ . Using that the form  $B_{n-1}$  is closed, we get

$$\begin{aligned} (-2i)^{n-2} \partial \bar{\partial} F^{n-2} &= \partial \bar{\partial} (B_{n-1} + \omega^{n\bar{n}})^{n-2} = \partial \bar{\partial} (B_{n-1})^{n-2} + (n-2) \partial \bar{\partial} ((B_{n-1})^{n-3} \wedge \omega^{n\bar{n}}) \\ &= (n-2) (B_{n-1})^{n-3} \wedge \partial \bar{\partial} (\omega^{n\bar{n}}) = -(n-2) (A_{n-1})^2 \wedge (B_{n-1})^{n-3}, \end{aligned}$$

where in the last equality we have used that  $\partial \bar{\partial} (\omega^{n\bar{n}}) = \bar{\partial} \omega^n \wedge \partial \omega^{\bar{n}} = -A_{n-1} \wedge A_{n-1}$ .

Therefore,  $F$  is astheno-Kähler if and only if  $(A_{n-1})^2 \wedge (B_{n-1})^{n-3} = 0$ .

We now use the balanced condition (6), i.e.  $a_{n-1} = -a_1 - \dots - a_{n-2}$ . Writing  $A_{n-1} = A_{n-2} + a_{n-1} \omega^{n-1\overline{n-1}}$  and  $B_{n-1} = B_{n-2} + b_{n-1} \omega^{n-1\overline{n-1}}$ , and noting that  $(A_{n-2})^2 \wedge (B_{n-2})^{n-3} = 0$ , one has that the astheno-Kähler condition is equivalent to

$$\begin{aligned} 0 &= (A_{n-1})^2 \wedge (B_{n-1})^{n-3} = (A_{n-2} + a_{n-1} \omega^{n-1\overline{n-1}})^2 \wedge (B_{n-2} + b_{n-1} \omega^{n-1\overline{n-1}})^{n-3} \\ &= \left[ (A_{n-2})^2 + 2a_{n-1} A_{n-2} \wedge \omega^{n-1\overline{n-1}} \right] \wedge \left[ (B_{n-2})^{n-3} + (n-3)b_{n-1} (B_{n-2})^{n-4} \wedge \omega^{n-1\overline{n-1}} \right] \\ &= [(n-3)b_{n-1} A_{n-2} - 2(a_1 + \dots + a_{n-2}) B_{n-2}] \wedge A_{n-2} \wedge (B_{n-2})^{n-4} \wedge \omega^{n-1\overline{n-1}}. \end{aligned}$$

Let us observe that in order to simplify this equation one can take  $a_1, \dots, a_{n-2} > 0$  and  $b_j = a_j$  for  $1 \leq j \leq n-2$ . Indeed, in this case we have that  $B_{n-2} = A_{n-2}$ , so it is enough to choose  $b_{n-1} = \frac{2}{n-3}(a_1 + \dots + a_{n-2})$  to get an astheno-Kähler metric  $F$  given by (7).

Finally, by Corollary 2.3 the metric  $F$  is in addition  $k$ -th Gauduchon for any  $1 \leq k \leq n-1$ . We notice that it can be directly proved that these complex nilmanifolds do not admit any SKT metric. Let us also note that the canonical bundle is holomorphically trivial, since the  $(n, 0)$ -form  $\Omega = \omega^{1 \cdots n}$  is closed.  $\square$

**Remark 2.5.** The (real) nilmanifolds in (5) correspond to the Lie algebras  $\mathfrak{g} = \mathfrak{h}_{2n+1} \times \mathbb{R}$ , where  $\mathfrak{h}_{2n+1}$  is the  $(2n+1)$ -dimensional Heisenberg algebra. Andrada, Barberis and Dotti proved in [2, Proposition 2.2] that every invariant complex structure  $J$  on these nilmanifolds is *abelian*, i.e.  $[Jx, Jy] = [x, y]$  for any  $x, y \in \mathfrak{g}$ . Moreover, there are exactly  $\left[\frac{n}{2}\right] + 1$  complex structures up to isomorphism. Let  $J_0$  be the complex structure defined by taking all the coefficients  $a_j$  positive numbers, i.e.  $a_1, \dots, a_{n-1} > 0$  in (5). One can prove the following result: *for any  $J$  not isomorphic to  $J_0$ , the complex nilmanifold admits a balanced metric and an astheno-Kähler metric which is  $k$ -th Gauduchon for any  $k$ .*

**Remark 2.6.** The complex structure in the 4-dimensional example given in [4] as well as those given in (5) are all abelian. Here we present a more general family of 4-dimensional complex nilmanifolds where the complex structure is not of that special type. Let us consider the complex structure equations

$$(8) \quad d\omega^1 = d\omega^2 = d\omega^3 = 0, \quad d\omega^4 = A\omega^{12} + B\omega^{13} + C\omega^{23} + \omega^{1\bar{1}} + \omega^{2\bar{2}} - 2\omega^{3\bar{3}},$$

where we require the coefficients  $A, B, C$  to belong to  $\mathbb{Q}(i)$  in order to ensure the existence of a lattice, so that equations (8) define a complex nilmanifold. Consider a metric  $F_{\alpha, \beta, \gamma}$  of the form

$$F_{\alpha, \beta, \gamma} = \frac{i}{2}(\alpha\omega^{1\bar{1}} + \beta\omega^{2\bar{2}} + \gamma\omega^{3\bar{3}} + \omega^{4\bar{4}}),$$

with  $\alpha, \beta, \gamma \in \mathbb{R}^+$ . On the one hand, it is easy to see that  $\alpha = \beta = \gamma = 1$  provides a balanced metric. On the other hand, the astheno-Kähler condition is satisfied if and only if  $\gamma = \frac{\alpha(|C|^2+4)+\beta(|B|^2+4)}{2-|A|^2} > 0$ , so it suffices to take any complex structure in (8) with  $|A| < \sqrt{2}$ . This provides a family of 4-dimensional complex nilmanifolds  $X_{A,B,C}$  with balanced and astheno-Kähler metrics which are  $k$ -th Gauduchon for any  $k$ . Notice that if  $(A, B, C) \neq (0, 0, 0)$ , then the Lie algebra underlying  $X_{A,B,C}$  is not isomorphic to  $\mathfrak{h}_7 \times \mathbb{R}$ .

#### ACKNOWLEDGMENTS

This work has been partially supported by the projects MINECO (Spain) MTM2014-58616-P and Gobierno de Aragón/Fondo Social Europeo, grupo consolidado E15-Geometría. Adela Latorre is also supported by a DGA predoctoral scholarship. We would like to thank Anna Fino for useful comments on the subject. We also thank the referee for comments and suggestions that have helped us to improve the final version of the paper.

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